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1996 J. Phys. A: Math. Gen. 29 L241

(http://iopscience.iop.org/0305-4470/29/10/004)

J. Phys. A: Math. Gen. **29** (1996) L241–L248. Printed in the UK

LETTER TO THE EDITOR

On the nonlinear Fock description of quantum systems with quadratic spectra

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Received 28 February 1996

Abstract. The nonlinear version of the Fock approach, developed previously for systems with (extended) equidistant spectra (Eleonsky V M and Korolev V G 1995 *J. Phys. A: Math. Gen.* **28** 4973), is also shown to describe systems with quadratic spectra. Some general properties of shift operators for such spectra are studied.

It is found, in particular, that families of shift operators related to the generators of the algebra $su(1, 1)$ as well as supersymmetry ladder operators arise naturally in that approach in a unified way.

In [1–3] the nonlinear generalization of the Fock approach [4] (based on introducing analogues of number operators nonlinear in the Hamiltonian as well as analogues of shift operators nonlinear in momentum) was presented. It allows one to investigate a wide range of one-dimensional potentials of the Schrödinger problem with prescribed strictly discrete spectra. The structure of the spectra was determined as one or a combination of several equidistant subsequences bounded below (and, possibly, above) and shifted arbitrarily relative to each other. As a particular case, equidistant spectra with a gap of given size and location were studied in detail.

In the present letter we demonstrate the possibility of an analogous approach to the Schrödinger problem for systems with quadratic spectra. As an example, that approach is used for a description of the system with the Pöschl–Teller potential [5]. It has been revealed that the spectrum shift operators, which arise traditionally as the $su(1, 1)$ generators (see, for example, [6] and references therein) as well as the supersymmetry (SUSY) ladder operators for the same model can be obtained in the framework of the present method as solutions of one and the same basic equation (1).

Previously in [8] the statement was made that solutions of the operator equation

$$[[H, L], L] = cL^2 \qquad c = \text{constant} \tag{1}$$

(H is a Hamiltonian) include a class of quantum models characterized by a constant second-order difference of the eigenvalues of H:

$$E_{n+2} - 2E_{n+1} + E_n = c \tag{2}$$

(i.e. having the quadratic spectrum $E_n \sim (n + n_0)^2$) much as the equation $[H, L] = \omega L$ defines a class of models related to the constant first-order difference $E_{n+1} - E_n = \omega$.

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0305-4470/96/100241+08$19.50

However, whereas in the latter case the operator L is always a shift operator of the solutions (not necessarily eigenstates) of the Schrödinger equation, the action of L obeying (1) on a solution of the Schrödinger problem may be more complicated.

Define the action of L on the eigenelements $(\psi_n(x), E_n)$ of the Schrödinger problem for the Hamiltonian H by the expression

$$L\psi_n(x) = \sum_k C_{n,k}\psi_k(x) \tag{3}$$

Substitution of that expression into (1) leads to the following system of equations:

$$\sum_k C_{n,k}C_{k,n'}\ (E_{n'} - 2E_k + E_n - c) = 0 \qquad n' = 0, 1, \ldots \tag{4}$$

which is an analogue of the well known selection rules $(E_{n'} - E_n - \omega)C_{k,n'} = 0$, originated by the linear operator equation $[H, L] = \omega L$.

• Let $C_{n,k} \neq 0$ only for $k = n + 1$ (in this case L is a shift operator). Then (4) takes the form

$$C_{n,n+1}C_{n+1,n'}\ (E_{n'} - 2E_{n+1} + E_n - c) = 0 \qquad n' = 0, 1, \ldots \tag{5}$$

and is non-trivial only for $n' = n + 2$, taking the form of condition (2).

• Now let L transform the wavefunction $\psi_n(x)$ into a pair of neighbouring eigenfunctions, i.e. $C_{n,k} \neq 0$ for $k = n \pm 1$. In that case (4) takes the form

$$C_{n,n-1}C_{n-1,n'}\ (E_{n'} - 2E_{n-1} + E_n - c) + C_{n,n+1}C_{n+1,n'}\ (E_{n'} - 2E_{n+1} + E_n - c) = 0$$
$$n' = 0, 1, \ldots \tag{6}$$

and leads to non-trivial conditions in the cases $n' = n \pm 2$ and $n' = n$. The first two of them lead to the invariance condition for the second-order difference (2); in the case $n' = n$ we get

$$C_{n,n-1}C_{n-1,n}\ (2E_n - 2E_{n-1} - c) = C_{n+1,n}C_{n,n+1}\ (2E_{n+1} - 2E_n + c). \tag{7}$$

Combining this with condition (2), rewritten as $2(E_{n+1} - E_n) - c = 2(E_n - E_{n-1}) + c$, we find that the expression

$$\mathcal{I}_n \equiv C_{n,n+1}C_{n+1,n}\left\{4(E_{n+1} - E_n)^2 - c^2\right\} \tag{8}$$

is a *spectrum invariant* in the sense that $\mathcal{I}_n = \text{constant}\ \forall\ n$.

• Finally, let L transform the wavefunction $\psi_n(x)$ into a linear combination of three neighbouring wavefunctions, i.e. $C_{n,k} \neq 0$ for $k = n - 1, n, n + 1$. In that case equation (4) takes the form

$$C_{n,n-1}C_{n-1,n'}\ (E_{n'} - 2E_{n-1} + E_n - c) + C_{n,n}C_{n,n'}\ (E'_n - E_n - c)$$
$$+C_{n,n+1}C_{n+1,n'}\ (E_{n'} - 2E_{n+1} + E_n - c) = 0 \qquad n' = 0, 1, \ldots \tag{9}$$

and leads to non-trivial conditions in five cases: $n' = n \pm 2$, $n' = n \pm 1$, and $n' = n$. The first two cases lead to condition (2); the next pair leads to the same condition

$$C_{n,n}\ (E_{n+1} - E_n - c) = C_{n+1,n+1}\ (E_{n+1} - E_n + c). \tag{10}$$

Finally, in the case $n' = n$ we obtain the relation

$$C_{n,n-1}C_{n-1,n}\ (2E_n - 2E_{n-1} - c) + C_{n+1,n}C_{n,n+1}\ (2E_n - 2E_{n+1} - c) = c\ C_{n,n}^2 \tag{11}$$

which is a generalization of the spectrum invariant (8).

These three cases (the last of them includes, obviously, the two previous particular cases) seem to exhaust all representations of L in the form (3) such that condition (2) is not violated (which is also consistent with the analysis of quadratic algebras in [9, 10]).

Consider a representation of (1) in the decomposed form

$$[H, L] = L\Omega \qquad [\Omega, L] = cL. \tag{12}$$

Here Ω is an arbitrary (and, generally, not self-adjoint) operator; existence of the operator inverse to L is assumed. Let (ψ, E) be some solution of the Schrödinger problem for the operator H: $H\psi = E\psi$. Then the function $\psi' \equiv L\psi$ is a solution of the Schrödinger problem with the operator $H' \equiv H - \Omega$, corresponding to the value $E' = E - c$:

$$(H - \Omega)\psi' = (E - c)\psi' \qquad \psi' \equiv L\psi. \tag{13}$$

Thus, in the general case L transforms solutions of the Schrödinger problem with the operator H into solutions of the Schrödinger problem with another operator H (possibly, not self-adjoint). If $\Omega = \Omega^\dagger$ then $L^\dagger$ transforms solutions of the Schrödinger equation with the operator H into solutions of the Schrödinger equation with the operator $H' = H + \Omega$ and value $E' = E$. In this case $LL^\dagger\psi$ and $L^\dagger L\psi$ again obey the original Schrödinger equation with the operator H and the initial value of the parameter E.

Consider in detail the following class of characteristic operators: $\Omega = \Omega(H)$. In that case L translates the solution ψ of the equation $H\psi = E\psi$ into the function $\psi' \equiv L\psi$ being the solution of the equation $H\psi' = E'\psi'$. From (12) it follows that

$$E' = E + \Omega(E) \qquad \Omega(E') - \Omega(E) = c. \tag{14}$$

(It is easy to verify that equation (2) is satisfied for any Ω.) Excluding E' we get the functional equation $\Omega(E + \Omega(E)) - \Omega(E) = c$, which can be set equal to the operator equation for $\Omega(H)$:

$$\Omega(H + \Omega(H)) - \Omega(H) = c \tag{15}$$

(the latter can also be obtained directly from (12)). Writing an analogue of (12) for $L^\dagger$, we arrive at the second functional relation for $\Omega(H)$:

$$\Omega(H - \Omega(H) + c) - \Omega(H) = -c. \tag{16}$$

Thus, the first of equations (12) supplemented by the conditions (15), (16) for $\Omega(H)$ leads to models related to the invariance condition for the second-order difference.

Let us find a solution of the functional equations (15), (16). From (15) we get a difference equation for the inverse function $H(\Omega)$:

$$H(\Omega + c) - H(\Omega) = \Omega. \tag{17}$$

Its solution is

$$H = \frac{(\Omega - c/2)^2}{2c} + F(\Omega) + H_0 \qquad H_0 = \text{constant} \tag{18}$$

where F is an arbitrary periodic function with period c: $F(\Omega+c) = F(\Omega)$. Let $F = H_0 \equiv 0$, $c > 0$. Then the particular solution of (15), (16) is

$$\Omega(H) = \frac{c}{2} \pm \sqrt{2cH} \qquad E > c/2. \tag{19}$$

Then the decomposed system (12) takes the form

$$[H, L] = L\left(\frac{c}{2} \pm \sqrt{2cH}\right) \tag{20}$$

$$[\sqrt{H}, L] = \pm\sqrt{c/2}\; L. \tag{21}$$

In this case the functional dependence between equations (12) expressed by condition (15) means that one can obtain equation (20) from (21), but the reverse is not true; in other words, under such a choice of $\Omega(H)$ the original equation (1) is directly reduced to (21). The latter equation can be treated as a formal analogue of the equation $[H, L] = \omega L$ related to the condition $E_{n'} - E_n = \omega$ [8, 1–3].

Indeed, the mapping $E \Rightarrow E'$ generated by equation (21) has the form

$$\sqrt{E'} - \sqrt{E} = \pm\sqrt{c/2} = \text{constant} \qquad E > c/2 \tag{22}$$

and realizes the invariance condition for the second-order difference (2) if executed at eigenelements of the Schrödinger problem.

On the other hand, equation (20) is more general than (21), so it can also describe systems with spectra that do not obey (2). It is easy to show that, apart from the two above monotonic sequences $\sqrt{E'} = \sqrt{E} \pm \sqrt{c/2}$, $E > c/2$, equation (20) admits the sequence $\sqrt{E'} = \sqrt{c/2} - \sqrt{E}$, $E < c/2$, which describes a cycle of two possible energy values. This points to the possibility (characteristic of the supersymmetry approaches) of describing in the framework of the present method systems with spectra consisting of both an infinite and a finite number of elements.

We come now to the quantum model obeying (20). In the one-dimensional case this equation is solvable for the operators L of the form $L(p, x) = L_{1/2}(x)\sqrt{H} + L_1(x)p$. There arises the Hamiltonian H with the Pöschl–Teller potential

$$H = \frac{1}{2}p^2 + \frac{l(l-1)}{\cos^2\xi} \tag{23}$$

and the operator L

$$L_{(\pm)} = \mp\sqrt{2}\sin\xi\ \sqrt{H} + \mathrm{i}\sqrt{\alpha}\cos\xi\ p \tag{24}$$

(hereafter we use the notation $\alpha \equiv c/2$, $\xi \equiv \sqrt{c}\,x$; $p = -\mathrm{i}\,\mathrm{d}/\mathrm{d}x$). Note that both the analysis of the problem on the basis of the algebra $su(1, 1)$ representations (see, for example, the summary tables in [6]) and another, less traditional approach shown in [7], lead to the same result.

The eigenelements of the corresponding Schrödinger problem are

$$\psi_n(x) = \cos^l\xi\ C_n^{(\nu)}(\sin\xi) \qquad E_n = \alpha(n+l)^2 \tag{25}$$

where $C_n^{(\nu)}(z)$ are the ultraspherical (Gegenbauer) polynomials (for brevity we use unnormalized wavefunctions).

The action of $L_{(\pm)}$, $L_{(\pm)}^{\dagger}$ on the eigenfunctions is determined as follows:

$$L_{(+)}\psi_n = -\sqrt{\alpha}(n+1)\ \psi_{n+1} \tag{26}$$

$$L_{(+)}^{\dagger}\psi_n = -\sqrt{\alpha}\left(n + 2(l-1) - \frac{l-1}{n+l}\right)\ \psi_{n-1} \tag{27}$$

$$L_{(-)}\psi_n = +\sqrt{\alpha}(n+2l+1)\ \psi_{n-1} \tag{28}$$

$$L_{(-)}^{\dagger}\psi_n = +\sqrt{\alpha}\left(n + 2 - \frac{l-1}{n+l}\right)\ \psi_{n+1}. \tag{29}$$

Thus, their action is reduced to a shift along the spectrum; the operators $L_{(+)}$, $L_{(-)}^{\dagger}$ are up-shift operators (in index n) whereas $L_{(-)}$, $L_{(+)}^{\dagger}$ are down-shift operators. We can introduce

one-parametric families (linear envelopes) of operators shifting up ($L_u^{(\gamma)}$) and down ($L_u^{(\gamma)}$):

$$L_u^{(\gamma)} = L_{(+)} + \gamma L_{(-)}^{\dagger} \tag{30}$$

$$L_d^{(\gamma)} = L_{(-)} + \gamma L_{(+)}^{\dagger}. \tag{31}$$

In the symmetric case ($\gamma = 1$) these operators take a particularly simple form:

$$L_u^{(1)} = L_{(+)}^S \qquad L_d^{(1)} = L_{(-)}^S \qquad L_{(\pm)}^S \equiv \sqrt{\alpha}\sin\xi \pm [\sqrt{H}, \sin\xi] \tag{32}$$

and their action is determined by the following expression:

$$L_{(\pm)}^S \psi_n = \sqrt{\alpha}\left\{1 \pm \frac{l-1}{n+l}\right\}\psi_{n\pm 1}. \tag{33}$$

Following Fock's approach [4], we consider *analogues of the number (of quanta) operators* $N_{(\pm)} \equiv L_{(\pm)}L_{(\pm)}^{\dagger}$, $\tilde{N}(\pm) \equiv L_{(\pm)}^{\dagger}L_{(\pm)}$. They are *not polynomial* in H (unlike the number operators considered in [1–3]) and are determined as follows:

$$N_{(\pm)} = \frac{(\pm\sqrt{H} - \sqrt{\alpha})(H \mp \sqrt{\alpha H} - U_0)}{\pm\sqrt{H}} \tag{34}$$

$$\tilde{N}(\pm) = \frac{\pm\sqrt{H}(H \pm \sqrt{\alpha H} - U_0)}{\sqrt{\alpha} \pm \sqrt{H}} \qquad U_0 \equiv l(l-1). \tag{35}$$

It is easy to verify that these operators satisfy both the general relation $[N, L] = L(\tilde{N} - N)$ used in [1–3] and its particular case $\tilde{N}(H) = N(\Omega + H)$ for the reduced equations (12); for this purpose the following easily proved equality can be used:

$$\left[\frac{1}{\sqrt{H}}, L\right] = -L\frac{\sqrt{\alpha}}{\sqrt{H}(\sqrt{H} + \sqrt{\alpha})}. \tag{36}$$

As can be seen from (34), for the system with the Hamiltonian (23) the analogue of the number operator N can be rewritten in the form

$$N = \sum_{k=-1}^{2} h_k \left(\sqrt{H}\right)^k = H \mp 2\sqrt{\alpha H} - (U_0 - \alpha) \pm \sqrt{\alpha}\frac{U_0}{\sqrt{H}}. \tag{37}$$

Then, using the relation $\tilde{N}(H) = N(\Omega + H)$ we get $\tilde{N} = \sum_{k=-1}^{2} h_k \left(\sqrt{\Omega + H}\right)^k$. The form of the characteristic operator $\Omega(H) = \alpha + 2\sqrt{\alpha H}$ allows us to remove the common radical sign in the above expression ($\sqrt{\Omega + H} \equiv \sqrt{\alpha} + \sqrt{H}$) and to arrive again at (35).

The expressions for the number operators defined on the base of the *symmetric* combined operators $L_{(\pm)}^S$ also have the simple form

$$N_{(\pm)}^S = \tilde{N}S_{(\mp)} \equiv L_{(+)}^S L^{S\dagger}{}_{(+)} = \alpha + \frac{\alpha U_0}{\pm\sqrt{H}(\sqrt{\alpha} \mp \sqrt{H})}. \tag{38}$$

Note also that the operator $L_{(+)}L_{(-)}$ is self-adjont:

$$L_{(+)}L_{(-)} = L_{(-)}^{\dagger}L_{(+)}^{\dagger} = -H + \sqrt{\alpha H} + U_0. \tag{39}$$

Following the generalized Fock approach [1–3] we define the initial element for the mapping of eigenelements of the Schrödinger problem by the condition for the eigenvalue ν of the number operator N to vanish. The eigenvalue of the operator $N_{(+)}$ (as well as the eigenvalue of the operator $N_{(+)}^S$) vanishes at the only point $E = \alpha l^2$; this value, in accordance with (25), corresponds to the ground state of the system. The L-mapping constructed on that state using the operator (24) generates an unbounded quadratic sequence E_n exhausting

the energy spectrum of the system. Since $l > 1$, all the points of that sequence lie in the region $E > \alpha \equiv c/2$ that corresponds to sequences preserving the second-order difference.

The case $c < 0$, related to the 'soliton' potential of finite depth and to the quadratic spectrum with a finite number of levels, is described analogously.

The question arises of whether it is possible to extend the class of quantum models using, as in [8, 1–3], operators L in the form of polynomials (in the momentum) whose coefficients are functions of the operator $\sqrt{H}$? It turns out that the search for solutions to (20) on the assumption that L has the form

$$L = \sum_{m=0}^{M} h_m(x)\ \left(\sqrt{H}\right)^m + \sum_{m=0}^{M-1} f_m(x)\ p\ \left(\sqrt{H}\right)^m$$

for the case $M = 3$ again leads to the same system with H of the form (23) and

$$L = L^{(0)}\left(1 + \beta\sqrt{H} + \gamma H\right) \qquad \beta, \gamma \in \mathcal{R}$$

where $L^{(0)}$ is the 'basic' operator (24). This result appears quite natural if we take into account the following evident property of equations (12): if some operator L is a solution then any operator of the form $L\mathcal{F}(H)$ is also a solution.

The above-considered operator L of the form (24) contained the operator $\sqrt{H}$ and was related to the characteristic operator $\Omega(H)$ of the form (19). Let us abandon the condition for Ω to be self-adjoint and construct the operator L as a solution of equation (1) (or of the system (12)) in the form of a polynomial of degree M in the momentum p only. Unlike the equation $[H, L] = \omega L,\ \omega =$ constant [8], even for $M = 2$ equation (1) leads to a substantial overdetermination of a system of equations for the coefficients of that polynomial.

In the case $M = 1$ we get the following system of equations for the potential $U(x)$ and coefficients of the polynomial $L = L_0(x) + L_1(x)\ p$:

$$L_1L_1'' - 2(L')^2 = L_1^2 \qquad L_1L_0'' - 3L_0'L_1' - 2L_1L_0 = 0$$

$$L_1(L_1U')' = -\{(L_0')^2 + L_0^2\} - \mathrm{i}\{L_1''L_0' + L_1'L_0\}$$

where a prime denotes a derivative with respect to the variable $\xi:\ \xi = \sqrt{c}(x - x_0)$. The solution of that system is

$$L_1 = \frac{A}{\cos\xi} \qquad L_0 = \frac{m}{1 - \sin\xi} + \frac{l}{\cos^2\xi} \tag{40}$$

$$U(x) = -\frac{1}{A^2}\frac{\left(m^2 + ml + \mathrm{i}Am/2 + l^2/2\right) + \left(m^2 + ml + \mathrm{i}A(m + l)/2\right)\sin\xi}{\cos^2\xi}$$

$$+\frac{k}{A}\sin\xi + U_0. \tag{41}$$

Consider the particular case $m = -l$ (and let $A = -\mathrm{i}$ without loss of generality). Then we obtain

$$H^{(l)} = \frac{1}{2}p^2 + \frac{l(l-1)}{\cos^2\xi} \qquad L^{(l)} = -\frac{\mathrm{i}}{\cos\xi}\ p - l\ \frac{\sin\xi}{\cos^2\xi} \tag{42}$$

These operators satisfy equations (12) with the not self-adjoint operator $\Omega^{(l)}$:

$$\Omega^{(l)} = -\mathrm{i}\tan\xi\ p + \left(\frac{1}{2} - \frac{l}{\cos^2\xi}\right). \tag{43}$$

(Note that $L^{(l)}$ and $\Omega^{(l)}$ are linked by the relation $\Omega^{(l)} = \sin\xi\ L^{(l)} + (\frac{1}{2} - l)$.)

In accordance with what has been said above about the action of L when the characteristic operator Ω is not a function of the Hamiltonian, the operator $L^{(l)}$ (42) is not an operator of shift along the spectrum of one and the same operator H; its action is determined by the expression

$$\psi' \equiv L^{(l)}\psi_n^{(l)} = -\frac{2l}{\cos\xi}\ \psi_{n-1}^{(l+1)} \tag{44}$$

where ψ' is a solution of (13).

Note, however, that the operator $L^{(l)}$ can be written in the form

$$L^{(l)} \equiv \frac{1}{\cos\xi}\Lambda^{(l)} \qquad \Lambda^{(l)} = -\mathrm{i}\ p - l\tan\xi. \tag{45}$$

The operator $\Lambda^{(l)}$ (differing from $L^{(l)}$ only by a weight function) acts on the wavefunction $\psi_n^{(l)}$ as follows:

$$\Lambda^{(l)}\psi_n^{(l)} = -2l\ \psi_{n-1}^{(l+1)} \tag{46}$$

i.e. it transforms the nth state in the potential with the parameter l into the $(n-1)$th state in the potential with the parameter $l+1$. This operator arises in the SUSY approach to the description of such spectra (see, for example, the review [6]) as a ladder operator. Unlike $L^{(l)}$ obeying the main equation (1), the operator $\Lambda \equiv \Lambda^{(l)}$ satisfies the following equations:

$$\Lambda\Lambda^{\dagger} = 2H^{(l+1)} - l^2 \qquad \Lambda^{\dagger}\Lambda = 2H^{(l)} - l^2 \tag{47}$$

or, in closed form w.r.t. Λ: $[H^{(l)}, \Lambda] = -(l/\cos^2\xi)\Lambda$.

Thus, we have obtained two kinds of operators L as solutions of equation (1) for the same Hamiltonian (23). The first of them is represented by the combined operators (30), (31) that are linear envelopes of the operators $L_{(\pm)}$, $L_{(\pm)}^{\dagger}$ (24). These operators are linear functions of $\sqrt{H}$ and $p \equiv -\mathrm{i}\,\mathrm{d}/\mathrm{d}x$; in the decomposed system (12) they are associated with the characteristic operator $\Omega = \Omega(H)$ of the form (19). The above operators perform a shift along the spectrum of the same Hamiltonian; they are related to the generators of the algebra $su(1, 1)$ for this system.

The second type of operators is represented by the operator $L^{(l)}$ that is a linear function of p only; it corresponds to the not self-adjoint characteristic operator Ω (43). Up to a weight function $L^{(l)}$ is related to the ladder operator Λ, which arises in the SUSY approach and performs a shift both in the eigenstate number n and in the structure parameter l (two analogous classes of operators arise also in the case of the modified Pöschl–Teller potential; see, for example, [11]).

We have shown that for the operators of the first kind one can build analogues of number operators and use the generalized Fock approach [1–3] for the determination and description of non-equidistant spectra.

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